

The real parts of the nontrivial Riemann zeta function zeros

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ABSTRACT

This theorem is based on holomorphy of studied functions and the fact that nearby of a singularity point the imaginary part of the specific function can accept zero value.

The colored markers are:

- - assumption or a fact which is not proven at present;
- - the statement which requires additional attention;
- - statement which is proved earlier or clearly undestandable.

THEOREM

- The real parts of all the nontrivial Riemann zeta function zeros ρ are equal $Re(\rho) = \frac{1}{2}$.

PROOF:

- According to the functional equality [10, p. 22], [5, p. 8-11]:

$$\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \Gamma\left(\frac{1-s}{2}\right) \pi^{-\frac{1-s}{2}} \zeta(1-s), \quad Re(s) > 0 \quad (1)$$

$\zeta(s)$ - the Riemann zeta function, $\Gamma(s)$ - the Gamma function.

- From [5, p. 8-11] $\zeta(\bar{s}) = \overline{\zeta(s)}$, it means that $\forall \rho = \sigma + it: \zeta(\rho) = 0$ and $0 \leq \sigma \leq 1$ we have:

$$\zeta(\bar{\rho}) = \zeta(1 - \rho) = \zeta(1 - \bar{\rho}) = 0 \quad (2)$$

- From [11], [9, p. 128], [10, p. 45] we know that $\zeta(s)$ has no nontrivial zeros on the line $\sigma = 1$ and consequently on the line $\sigma = 0$ also, in accordance with (2) they don't exist.
- Let's denote the set of nontrivial zeros $\zeta(s)$ through \mathcal{P} (multiset with consideration of multiplicity):

$$\mathcal{P} \stackrel{\text{def}}{=} \{\rho : \zeta(\rho) = 0, \rho = \sigma + it, 0 < \sigma < 1\}.$$

$$\text{And: } \mathcal{P}_1 \stackrel{\text{def}}{=} \left\{ \rho : \zeta(\rho) = 0, \rho = \sigma + it, 0 < \sigma < \frac{1}{2} \right\}, \quad (3)$$

$$\mathcal{P}_2 \stackrel{\text{def}}{=} \left\{ \rho : \zeta(\rho) = 0, \rho = \frac{1}{2} + it \right\},$$

$$\mathcal{P}_3 \stackrel{\text{def}}{=} \left\{ \rho : \zeta(\rho) = 0, \rho = \sigma + it, \frac{1}{2} < \sigma < 1 \right\}.$$

Then:

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \quad \text{and} \quad \mathcal{P}_1 \cap \mathcal{P}_2 = \mathcal{P}_2 \cap \mathcal{P}_3 = \mathcal{P}_1 \cap \mathcal{P}_3 = \emptyset,$$

$$\mathcal{P}_1 = \emptyset \Leftrightarrow \mathcal{P}_3 = \emptyset.$$

- Hadamard's theorem (Weierstrass preparation theorem) on the decomposition of function through the roots gives us the following result [10, p. 30], [5, p. 31], [12]:

$$\zeta(s) = \frac{\pi^{\frac{s}{2}} e^{as}}{s(s-1)\Gamma\left(\frac{s}{2}\right)} \prod_{\rho \in \mathcal{P}} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}, \quad \text{Re}(s) > 0 \quad (4)$$

$$a = \ln 2\sqrt{\pi} - \frac{\gamma}{2} - 1, \quad \gamma - \text{Euler's constant and}$$

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2}\ln\pi + a - \frac{1}{s} + \frac{1}{1-s} - \frac{1}{2} \frac{\Gamma'(\frac{s}{2})}{\Gamma(\frac{s}{2})} + \sum_{\rho \in \mathcal{P}} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) \quad (5)$$

- According to the fact that $\frac{\Gamma'(\frac{s}{2})}{\Gamma(\frac{s}{2})}$ - Digamma function of [10, p. 31], [5, p. 23] we have:

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} + \sum_{\rho \in \mathcal{P}} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{s+2n} - \frac{1}{2n} \right) + C, \quad (6)$$

$$C = \text{const}$$

- From [4, p. 160], [8, p. 272], [3, p. 81]:

$$\sum_{\rho \in \mathcal{P}} \frac{1}{\rho} = 1 + \frac{\gamma}{2} - \ln 2\sqrt{\pi} = 0,0230957\dots \quad (7)$$

- Indeed, from (2):

$$\sum_{\rho \in \mathcal{P}} \frac{1}{\rho} = \frac{1}{2} \sum_{\rho \in \mathcal{P}} \left(\frac{1}{1-\rho} + \frac{1}{\rho} \right)$$

- From (5):

$$2 \sum_{\rho \in \mathcal{P}} \frac{1}{\rho} = \lim_{s \rightarrow 1} \left(\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{1-s} + \frac{1}{s} - a - \frac{1}{2}\ln\pi + \frac{1}{2} \frac{\Gamma'(\frac{s}{2})}{\Gamma(\frac{s}{2})} \right).$$

- Also it's known, for example, from [10, p. 49], [3, p. 98] that the number of nontrivial zeros of $\rho = \sigma + it$ in strip $0 < \sigma < 1$, the imaginary parts of which t are less than some number $T > 0$ is limited, i.e.

$$\| \{ \rho : \rho \in \mathcal{P}, \rho = \sigma + it, |t| < T \} \| < \infty.$$

- Indeed, it can be presented that on the contrary the sum of $\sum_{\rho \in \mathcal{P}} \frac{1}{\rho}$ would have been unlimited.

- Thus $\forall T > 0 \exists \delta_x > 0, \delta_y > 0$ such that

$$\text{in area } 0 < t \leq \delta_y, 0 < \sigma \leq \delta_x \text{ there are no zeros } \rho = \sigma + it \in \mathcal{P}. \quad (8)$$

Let's consider random root $q \in \mathcal{P}_1 \cup \mathcal{P}_2$

Let's denote $k(q)$ the multiplicity of the root q .

Let's examine the area $Q(R) \stackrel{\text{def}}{=} \{s : \|s - q\| \leq R, R > 0\}$.

- From the fact of finiteness of set of nontrivial zeros $\zeta(s)$ in the limited area follows $\exists R > 0$, such that $Q(R)$ does not contain any root from \mathcal{P} except q .

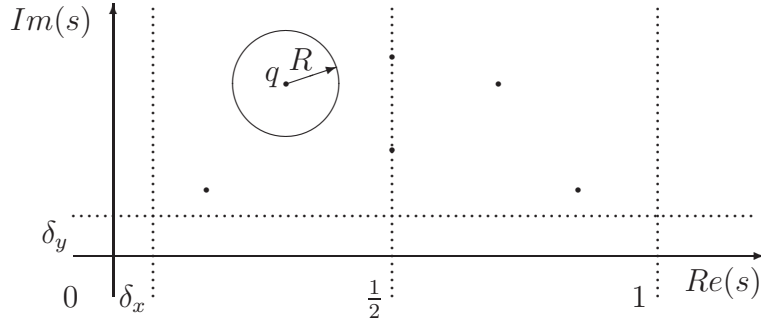


Fig. 1.

- From [1], [10, p. 31], [5, p. 23] we know that the Digamma function $\frac{\Gamma'(\frac{s}{2})}{\Gamma(\frac{s}{2})}$ in the area $Q(R)$ has no poles, i.e. $\forall s \in Q(R)$

$$\left\| \frac{\Gamma'(\frac{s}{2})}{\Gamma(\frac{s}{2})} \right\| < \infty.$$

Let's denote:

$$I_{\mathcal{P}}(s) \stackrel{\text{def}}{=} -\frac{1}{s} + \frac{1}{1-s} + \sum_{\rho \in \mathcal{P}} \frac{1}{s-\rho}$$

and

$$I_{\mathcal{P} \setminus \{q\}}(s) = -\frac{1}{s} + \frac{1}{1-s} + \sum_{\rho \in \mathcal{P} \setminus \{q\}} \frac{1}{s-\rho}. \quad (9)$$

Hereinafter $\mathcal{P} \setminus \{q\} \stackrel{\text{def}}{=} \mathcal{P} \setminus \{(q, k(q))\}$ (the difference in the multiset).

Summation - $\sum_{\rho \in \mathcal{P}} \frac{1}{s - \rho}$ and $\sum_{\rho \in \mathcal{P} \setminus \{q\}} \frac{1}{s - \rho}$ further we shall consider as the sum of pairs $\left(\frac{1}{s - \rho} + \frac{1}{s - (1 - \rho)} \right)$ and $\sum_{\rho \in \mathcal{P}} \frac{1}{\rho}$ as the sum of pairs $\left(\frac{1}{\rho} + \frac{1}{1 - \rho} \right)$ as a consequence of division of the sum from (6) $\sum_{\rho \in \mathcal{P}} \left(\frac{1}{s - \rho} + \frac{1}{\rho} \right)$ into $\sum_{\rho \in \mathcal{P}} \frac{1}{s - \rho} + \sum_{\rho \in \mathcal{P}} \frac{1}{\rho}$. As specified in [4], [6], [8], [10].

- Let's note that $I_{\mathcal{P} \setminus \{q\}}(s)$ is holomorphic function $\forall s \in Q(R)$.

Then from (5) we have:

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2} \ln \pi + a - \frac{1}{2} \frac{\Gamma' \left(\frac{s}{2} \right)}{\Gamma \left(\frac{s}{2} \right)} + \sum_{\rho \in \mathcal{P}} \frac{1}{\rho} + I_{\mathcal{P}}(s). \quad (10)$$

And in view of (7):

$$\text{Im} \frac{\zeta'(s)}{\zeta(s)} = \text{Im} \left(-\frac{1}{2} \frac{\Gamma' \left(\frac{s}{2} \right)}{\Gamma \left(\frac{s}{2} \right)} + I_{\mathcal{P}}(s) \right). \quad (11)$$

Let's note that from the equality of

$$\sum_{\rho \in \mathcal{P}} \frac{1}{1 - s - \rho} = - \sum_{(1-\rho) \in \mathcal{P}} \frac{1}{s - (1 - \rho)} = - \sum_{\rho \in \mathcal{P}} \frac{1}{s - \rho} \quad (12)$$

follows that:

$$I_{\mathcal{P}}(1 - s) = -I_{\mathcal{P}}(s), \quad I_{\mathcal{P} \setminus \{q\}}(1 - s) = -I_{\mathcal{P} \setminus \{1-q\}}(s), \quad \text{Re}(s) > 0.$$

- Besides

$$I_{\mathcal{P} \setminus \{q\}}(s) = I_{\mathcal{P}}(s) - \frac{k(q)}{s - q}$$

and $I_{\mathcal{P} \setminus \{q\}}(s)$ is limited in the area of $s \in Q(R)$ as a result of absence of its poles in this area as well as its differentiability in each point of this area.

- If in (5) to replace s with $1 - s$ that in view of (7):

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{\zeta'(1-s)}{\zeta(1-s)} = -\frac{1}{2} \frac{\Gamma'(\frac{s}{2})}{\Gamma(\frac{s}{2})} - \frac{1}{2} \frac{\Gamma'(\frac{1-s}{2})}{\Gamma(\frac{1-s}{2})} + \ln \pi, \quad \text{Re}(s) > 0. \quad (13)$$

- Let's examine a circle with the center in a point q and radius $r \leq R$, laying in the area of $Q(R)$:

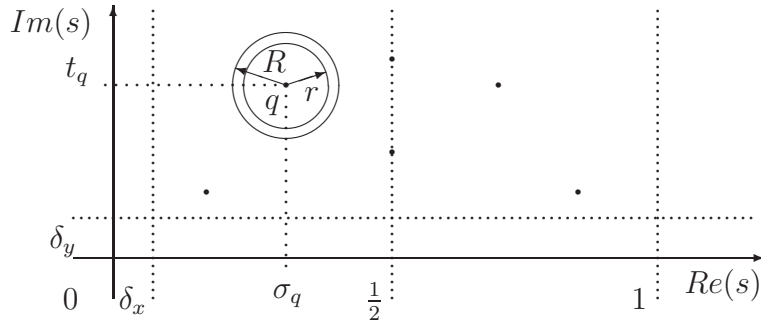


Fig. 2.

- For $s = x + iy$, $q = \sigma_q + it_q$

$$\text{Im} \frac{k(q)}{s - q} = \text{Im} \frac{k(q)}{x + iy - \sigma_q - it_q} = \frac{k(q)(t_q - y)}{(x - \sigma_q)^2 + (y - t_q)^2} = k(q) \frac{t_q - y}{r^2},$$

- Consider also the function $\ln \zeta(s)$ - principal branch function $\text{Ln} \zeta(s)$ for which of the (4), in view of (7) $\forall s \in Q(R)$ is true:

$$\ln \zeta(s) = \frac{\ln \pi}{2} s - \ln(s(1-s)) - \ln \Gamma\left(\frac{s}{2}\right) + \sum_{\rho \in \mathcal{P}} \ln \left(1 - \frac{s}{\rho}\right). \quad (14)$$

The sum as stipulated earlier, is taken in pairs:

$$\ln \left(1 - \frac{s}{\rho}\right) + \ln \left(1 - \frac{s}{1-\rho}\right).$$

Let's designate the real function of two variables for $s = x + iy$:

$$\omega(x, y) \stackrel{\text{def}}{=} \text{Im} \left(-\ln(s(1-s)) - \ln \Gamma\left(\frac{s}{2}\right) + \sum_{\rho \in \mathcal{P} \setminus \{q\}} \ln \left(1 - \frac{s}{\rho}\right) \right).$$

- Note that the function $\omega(x, y)$ and its partial derivatives on both variables exist and are limited to $\forall s = x + iy \in Q(R)$, since

$$\omega(x, y) = \text{Im} \ln \zeta(x + iy) - \frac{\ln \pi}{2} y - k(q) \text{Im} \ln \left(1 - \frac{x + iy}{q}\right).$$

Inside of area $Q(R)$ we take a point $M \stackrel{\text{def}}{=} (x_M, y_M)$, does not coincide with the point q .

- Let's draw a looped curve from the point M so that it doesn't pass through the point q and is described by the function which has a continuous derivative at each point.

Let's designate: $f_M(\tau) \stackrel{\text{def}}{=} f_x(\tau) + if_y(\tau)$:

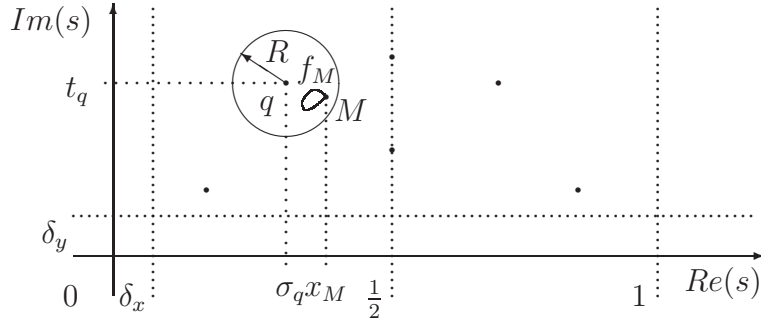


Fig. 3.

In accordance with the construction let's denote $\tau_{M,1}$ and $\tau_{M,2}$ such that:

$$f_M(\tau_{M,1}) = x_M + iy_M, \quad f_M(\tau_{M,2}) = x_M + iy_M.$$

- Function $Im \ln \zeta(s) - \frac{\ln \pi}{2} Im(s)$ is differentiable and therefore continuous and differentiable in τ function $Im \ln \zeta(f_M(\tau)) - \frac{\ln \pi}{2} Im(f_M(\tau))$.

It means that continuous on the segment and differentiable in the domestic range of this segment the real function gets at its ends the same values:

$$\begin{aligned} & Im \ln \zeta(f_M(\tau_{M,1})) - \frac{\ln \pi}{2} Im(f_M(\tau_{M,1})) = \\ & = Im \ln \zeta(f_M(\tau_{M,2})) - \frac{\ln \pi}{2} Im(f_M(\tau_{M,2})) = \\ & = Im \ln \zeta(f_M(x_M + iy_M)) - \frac{\ln \pi}{2} Im(f_M(x_M + iy_M)). \end{aligned}$$

- By Rolle's theorem on the extremum of a differentiable function on the interval we have:

$$\exists \tau_1 \in (\tau_{M,1}, \tau_{M,2}) : \left(Im \ln \zeta(f_M(\tau)) - \frac{\ln \pi}{2} Im(f_M(\tau)) \right)'_{\tau=\tau_1} = 0. \quad (15)$$

I.e. on a curve described by function $f_M(\tau)$, $\tau \in (\tau_{M,1}, \tau_{M,2})$ there is a point $\Theta_w = \Theta_w(M) \stackrel{\text{def}}{=} f_M(\tau_1)$ for which it is true (15).

- Let's consider the following option line, as a closed curve which passes through a point of M .

For any $0 < r < R$ construct a circle centered at the point q and the radius r .

The point of intersection of the left semicircle of the circle and the line $y - t_q = x - \sigma_q$ let's denote as $J \stackrel{\text{def}}{=} (x_J, y_J)$.

Let's construct a circle with the center in a point J , with radius:

$$0 < r_\delta < \min(r, R - r).$$

As a point of M let's take more distant from q point of intersection of the circle with the line $y - t_q = x - \sigma_q$.

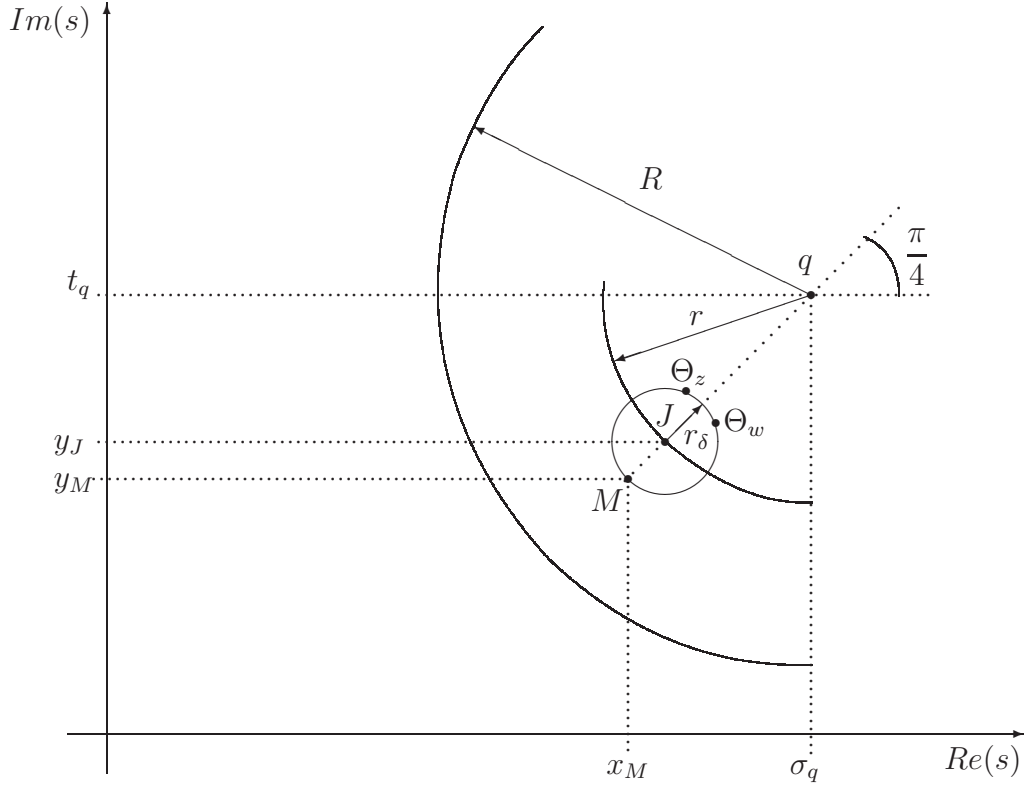


Fig. 4.

- As the desired curve, we consider the circle with center at J and the radius r_δ .

Let's notice that the received curve satisfies to all declared properties for any as much as small radius since $\min(r, R - r)$.

- Let's assign:

$$0 < r_\delta = O(r^3)_{r \rightarrow 0}. \quad (16)$$

Then $\exists 0 < R_1 \leq R : \forall 0 < r < R_1$ the condition is satisfied:

$$0 < r_\delta < \min(r, R - r).$$

As τ take $Im(s) = y$, then:

$$f_y(y) = y, \quad x = f_x(y) = x_J \pm \sqrt{r_\delta^2 - (y - y_J)^2}.$$

- According to [1, p. 67, 82]:

$$\begin{aligned} Im \ln \left(1 - \frac{x + iy}{q} \right) &= Im \ln (\sigma_q - x + i(t_q - y)) - Im \ln (q) = \\ &= \arctan \left(\frac{t_q - y}{\sigma_q - x} \right) - \arctan \left(\frac{t_q}{\sigma_q} \right), \end{aligned}$$

$$\frac{d}{dx} \arctan(x) = \frac{1}{1 + x^2}.$$

- Then:

$$\begin{aligned} \frac{d}{d\tau} Im \ln \left(1 - \frac{x(\tau) + i\tau}{q} \right)_{\tau=\tau_1} &= \frac{d}{d\tau} \arctan \left(\frac{t_q - \tau}{\sigma_q - x(\tau)} \right)_{\tau=\tau_1} = \\ &= \frac{1}{1 + \left(\frac{t_q - \tau_1}{\sigma_q - x(\tau_1)} \right)^2} \frac{\partial}{\partial \tau} \left(\frac{t_q - \tau}{\sigma_q - x(\tau)} \right)_{\tau=\tau_1} + \\ &+ \frac{1}{1 + \left(\frac{t_q - \tau_1}{\sigma_q - x(\tau_1)} \right)^2} \frac{\partial}{\partial x} \left(\frac{t_q - \tau_1}{\sigma_q - x} \right)_{x=x(\tau_1)} x'(\tau_1) = \\ &= -\frac{\sigma_q - x(\tau_1)}{(\sigma_q - x(\tau_1))^2 + (t_q - \tau_1)^2} + \frac{x'(\tau_1)(t_q - \tau_1)}{(\sigma_q - x(\tau_1))^2 + (t_q - \tau_1)^2}. \end{aligned} \quad (17)$$

- And for the $s = x + iy$ equation (15) at the point $\Theta_w = (x(\tau_1), \tau_1)$ can be written as follows:

$$\begin{aligned} & \frac{\sigma_q - x(\tau_1)}{(\sigma_q - x(\tau_1))^2 + (t_q - \tau_1)^2} - \frac{x'(\tau_1)(t_q - \tau_1)}{(\sigma_q - x(\tau_1))^2 + (t_q - \tau_1)^2} = \\ & = \omega(x(\tau_1), y)'_{y=\tau_1} + x(\tau_1)'\omega(x, \tau_1)'_{x=x(\tau_1)}. \end{aligned} \quad (18)$$

- At $r \rightarrow 0$ and (16):

$$(\sigma_q - x(\tau_1))^2 + (t_q - \tau_1)^2 \leq \left(\frac{\sqrt{2}}{2}r + r_\delta\right)^2 + \left(\frac{\sqrt{2}}{2}r + r_\delta\right)^2 = r^2 + O(r^4),$$

$$(\sigma_q - x(\tau_1))^2 + (t_q - \tau_1)^2 \geq \left(\frac{\sqrt{2}}{2}r - r_\delta\right)^2 + \left(\frac{\sqrt{2}}{2}r - r_\delta\right)^2 = r^2 + O(r^4),$$

i.e.

$$(\sigma_q - x(\tau_1))^2 + (t_q - \tau_1)^2 = r^2 + O(r^4).$$

- Hence from the equation (18) at $r \rightarrow 0$ $x'(\tau_1)$:

$$x'(\tau_1) = 1 - \omega(x(\tau_1), y)'_{y=\tau_1} \sqrt{2}r + O(r^2). \quad (19)$$

Similar reasonings for $s = x + iy$ in the same area $s \in Q(R)$ points q for function $Im \ln \zeta(1 - s) - \frac{\ln \pi}{2} Im(1 - s)$ we shall come to conclusion that on the circle with the center in J and radius r_δ there should be a point $\Theta_z = \Theta_z(M) \stackrel{\text{def}}{=} f_M(\tau_2)$ for some $\tau_2 \in (\tau_{M,1}, \tau_{M,2})$ for which equality is true:

$$(Im \ln \zeta(1 - f_M(\tau))) - \frac{\ln \pi}{2} Im(1 - f_M(\tau))'_{\tau=\tau_2} = 0. \quad (20)$$

- Indeed, if we apply the (14) value is $1 - s$ instead of s :

$$\begin{aligned} & \ln \zeta(1 - s) - \frac{\ln \pi}{2}(1 - s) = \\ & -\ln((1 - s)s) - \ln \Gamma\left(\frac{1 - s}{2}\right) + \sum_{\rho \in \mathcal{P}} \ln\left(1 - \frac{1 - s}{\rho}\right) \end{aligned}$$

that in view of that $\forall \rho \in \mathcal{P}$:

$$\left(1 - \frac{1 - s}{\rho}\right) \left(1 - \frac{1 - s}{1 - \rho}\right) = \left(1 - \frac{s}{\rho}\right) \left(1 - \frac{s}{1 - \rho}\right),$$

we have:

$$\ln \zeta(1 - s) - \frac{\ln \pi}{2}(1 - s) = -\ln(s(1 - s)) - \ln \Gamma\left(\frac{1 - s}{2}\right) + \sum_{\rho \in \mathcal{P}} \ln\left(1 - \frac{s}{\rho}\right). \quad (21)$$

Let's designate the real function of two variables for $s = x + iy$:

$$\omega(x, y)_- \stackrel{\text{def}}{=} \text{Im} \left(-\ln(s(1 - s)) - \ln \Gamma\left(\frac{1 - s}{2}\right) + \sum_{\rho \in \mathcal{P} \setminus \{q\}} \ln\left(1 - \frac{s}{\rho}\right) \right).$$

- Note that the function $\omega(x, y)_-$ and its partial derivatives on both variables exist and are limited to $\forall s = x + iy \in Q(R)$, since

$$\omega(x, y)_- = \text{Im} \ln \zeta(1 - x - iy) + \frac{\ln \pi}{2}y - k(q) \text{Im} \ln\left(1 - \frac{x + iy}{q}\right).$$

And all reasonings for functions $\ln \zeta(1 - s) - \frac{\ln \pi}{2}(1 - s)$ and $\omega(x, y)_-$ are similar to reasonings for appropriate functions $\ln \zeta(s) - \frac{\ln \pi}{2}(s)$ and $\omega(x, y)$.

- So, on the circle with the center in J and radius r_δ are two points of Θ_w and Θ_z , for which the equalities (15) and (20) respectively.

And similarly (19) at $r \rightarrow 0$:

$$x'(\tau_2) = 1 - \omega_-(x(\tau_2), y)'_{y=\tau_2} r + O(r^2). \quad (22)$$

For $\forall s : \|s - (x_J + iy_J)\| < r_\delta$ derivative of function $-\ln(s(1-s)) + \sum_{\rho \in \mathcal{P}} \ln\left(1 - \frac{s}{\rho}\right)$ is continuous and equal to:

$$\frac{d}{ds} \left(-\ln(s(1-s)) + \sum_{\rho \in \mathcal{P}} \ln\left(1 - \frac{s}{\rho}\right) \right) = I_{\mathcal{P}}(s).$$

Hence for fixed $0 < r < R_1$ from the continuity of the given above function at the point J :

$\forall \varepsilon_r > 0, \exists \delta_{\varepsilon_r} > 0 : \forall s : \|s - (x_J + iy_J)\| < \delta_{\varepsilon_r}$ follows:

$$|ReI_{\mathcal{P}}(s) - ReI_{\mathcal{P}}(x_J + iy_J)| < \frac{\varepsilon_r}{4},$$

$$|ImI_{\mathcal{P}}(s) - ImI_{\mathcal{P}}(x_J + iy_J)| < \frac{\varepsilon_r}{4}.$$

- Let's assign $\varepsilon_r = r$ then $\forall 0 < r_\delta < \delta_{\varepsilon_r}$, at $r \rightarrow 0$, with the condition (16) similar to (17) where $f_y(\tau) = \tau$, $f_x(\tau) = x(\tau)$ is true:

$$\begin{aligned} & \frac{d}{d\tau} Im \left(-\ln(f_M(\tau)(1-f_M(\tau))) + \sum_{\rho \in \mathcal{P}} \ln\left(1 - \frac{f_M(\tau)}{\rho}\right) \right)_{\tau=\tau_1} - \\ & - \frac{d}{d\tau} Im \left(-\ln(f_M(\tau)(1-f_M(\tau))) + \sum_{\rho \in \mathcal{P}} \ln\left(1 - \frac{f_M(\tau)}{\rho}\right) \right)_{\tau=\tau_2} = \\ & = (-ReI_{\mathcal{P}}(\Theta_w) + x(\tau_1)'ImI_{\mathcal{P}}(\Theta_w)) - \\ & - (-ReI_{\mathcal{P}}(\Theta_z) + x(\tau_2)'ImI_{\mathcal{P}}(\Theta_z)) = \end{aligned}$$

$$\begin{aligned}
&= -ReI_{\mathcal{P}}(\Theta_w) + ReI_{\mathcal{P}}(x_J + iy_J) - ReI_{\mathcal{P}}(x_J + iy_J) + ReI_{\mathcal{P}}(\Theta_z) + \\
&\quad + x(\tau_1)' ImI_{\mathcal{P}}(\Theta_w) - x(\tau_1)' ImI_{\mathcal{P}}(x_J + iy_J) + \\
&\quad + x(\tau_2)' ImI_{\mathcal{P}}(x_J + iy_J) - x(\tau_2)' ImI_{\mathcal{P}}(\Theta_z) + \\
&\quad + x(\tau_1)' ImI_{\mathcal{P}}(x_J + iy_J) - x(\tau_2)' ImI_{\mathcal{P}}(x_J + iy_J) = \\
&\quad = O\left(\frac{r}{4} + \frac{r}{4} + \frac{r}{4} |x(\tau_1)'| + \frac{r}{4} |x(\tau_2)'|\right) + \\
&\quad + (x(\tau_1)' - x(\tau_2)') ImI_{\mathcal{P}}(x_J + iy_J) = \\
&\quad = (x(\tau_1)' - x(\tau_2)') ImI_{\mathcal{P}}(x_J + iy_J) + O(r) = \\
&\quad = (\omega_-(x(\tau_2), y)'_{y=\tau_2} \sqrt{2}r - \omega(x(\tau_1), y)'_{y=\tau_1} \sqrt{2}r + O(r^2)) * \\
&\quad * \left(ImI_{\mathcal{P} \setminus \{q\}}(x_J + iy_J) + \frac{t_q - y_J}{(\sigma_q - x_J)^2 + (t_q - y_J)^2} \right) + O(r) = \\
&\quad = \omega_-(x(\tau_2), y)'_{y=\tau_2} - \omega(x(\tau_1), y)'_{y=\tau_1} + O(r) = \\
&\quad = -\frac{\partial}{\partial y} Im \ln \Gamma \left(\frac{1 - x(\tau_2) - iy}{2} \right)_{y=\tau_2} + \frac{\partial}{\partial y} Im \ln \Gamma \left(\frac{x(\tau_1) + iy}{2} \right)_{y=\tau_1} + \\
&\quad + ReI_{\mathcal{P} \setminus \{q\}}(\Theta_w) - ReI_{\mathcal{P} \setminus \{q\}}(\Theta_z) + O(r) = \\
&\quad = -\frac{\partial}{\partial y} Im \ln \Gamma \left(\frac{1 - x(\tau_2) - iy}{2} \right)_{y=\tau_2} + \frac{\partial}{\partial y} Im \ln \Gamma \left(\frac{x(\tau_1) + iy}{2} \right)_{y=\tau_1} + \\
&\quad + o(1). \tag{23}
\end{aligned}$$

- Let's consider a difference of the equations (15) and (20) which on construction is equal 0 for $\forall 0 < r < R_1$:

$$\begin{aligned}
\lim_{r \rightarrow 0} 0 &= \lim_{r \rightarrow 0} \left(\left(\operatorname{Im} \ln \zeta(f_M(\tau)) - \frac{\ln \pi}{2} \operatorname{Im}(f_M(\tau)) \right)'_{\tau=\tau_1} - \right. \\
&\quad \left. - \left(\operatorname{Im} \ln \zeta(1 - f_M(\tau)) - \frac{\ln \pi}{2} \operatorname{Im}(1 - f_M(\tau)) \right)'_{\tau=\tau_2} \right) = \\
&= \lim_{r \rightarrow 0} \left(- \left(\operatorname{Im} \ln \Gamma \left(\frac{f_M(\tau)}{2} \right) \right)'_{\tau=\tau_1} + \left(\operatorname{Im} \ln \Gamma \left(\frac{1 - f_M(\tau)}{2} \right) \right)'_{\tau=\tau_2} + \right. \\
&\quad \left. + \frac{d}{d\tau} \operatorname{Im} \left(-\ln(f_M(\tau)(1 - f_M(\tau))) + \sum_{\rho \in \mathcal{P}} \ln \left(1 - \frac{f_M(\tau)}{\rho} \right) \right)_{\tau=\tau_1} - \right. \\
&\quad \left. - \left(\frac{d}{d\tau} \operatorname{Im} \left(-\ln(f_M(\tau)(1 - f_M(\tau))) + \sum_{\rho \in \mathcal{P}} \ln \left(1 - \frac{f_M(\tau)}{\rho} \right) \right)_{\tau=\tau_2} \right) \right) =
\end{aligned}$$

And then from (23) follows:

$$\begin{aligned}
&= \lim_{r \rightarrow 0} \left(-x(\tau_1)' \frac{\partial}{\partial x} \left(\operatorname{Im} \ln \Gamma \left(\frac{x + i\tau_1}{2} \right) \right)_{x=x(\tau_1)} - \right. \\
&\quad \left. - \frac{\partial}{\partial y} \left(\operatorname{Im} \ln \Gamma \left(\frac{x(\tau_1) + iy}{2} \right) \right)_{y=\tau_1} + \right. \\
&\quad \left. + x(\tau_2)' \frac{\partial}{\partial x} \left(\operatorname{Im} \ln \Gamma \left(\frac{1 - x - i\tau_2}{2} \right) \right)_{x=x(\tau_2)} + \right. \\
&\quad \left. + \frac{\partial}{\partial y} \left(\operatorname{Im} \ln \Gamma \left(\frac{1 - x(\tau_2) - iy}{2} \right) \right)_{y=\tau_2} - \right. \\
&\quad \left. - \frac{\partial}{\partial y} \left(\operatorname{Im} \ln \Gamma \left(\frac{1 - x(\tau_2) - iy}{2} \right) \right)_{y=\tau_2} + \right. \\
&\quad \left. + \frac{\partial}{\partial y} \left(\operatorname{Im} \ln \Gamma \left(\frac{x(\tau_1) + iy}{2} \right) \right)_{y=\tau_1} + o(1) \right) =
\end{aligned}$$

$$\begin{aligned}
&= \lim_{r \rightarrow 0} \left(-\frac{\partial}{\partial x} \left(\operatorname{Im} \ln \Gamma \left(\frac{x + iy(\tau_1)}{2} \right) \right) \right)_{x=\tau_1} + \\
&\quad + \frac{\partial}{\partial x} \left(\operatorname{Im} \ln \Gamma \left(\frac{1-x - iy(\tau_2)}{2} \right) \right)_{x=\tau_2} = \\
&= \operatorname{Im} \left(-\frac{d}{ds} \ln \Gamma \left(\frac{s}{2} \right)_{s=q} + \frac{d}{ds} \ln \Gamma \left(\frac{1-s}{2} \right)_{s=q} \right) = \\
&= \operatorname{Im} \left(-\frac{1}{2} \frac{\Gamma' \left(\frac{q}{2} \right)}{\Gamma \left(\frac{q}{2} \right)} - \frac{1}{2} \frac{\Gamma' \left(\frac{1-q}{2} \right)}{\Gamma \left(\frac{1-q}{2} \right)} \right) = 0.
\end{aligned}$$

- Thus for the selected root q is:

$$\operatorname{Im} \left(-\frac{1}{2} \frac{\Gamma' \left(\frac{q}{2} \right)}{\Gamma \left(\frac{q}{2} \right)} - \frac{1}{2} \frac{\Gamma' \left(\frac{1-q}{2} \right)}{\Gamma \left(\frac{1-q}{2} \right)} \right) = 0, \quad \forall q \in \mathcal{P}_1 \cup \mathcal{P}_2. \quad (24)$$

- From (6) equality (24) can be rewritten as follows:

$$\sum_{n=0}^{\infty} \left(\frac{t_q}{(2n + \sigma_q)^2 + t_q^2} - \frac{t_q}{(2n + 1 - \sigma_q)^2 + t_q^2} \right) = 0.$$

I.e.

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{t_q((2n + 1 - \sigma_q)^2 - (2n + \sigma_q)^2)}{((2n + \sigma_q)^2 + t_q^2)((2n + 1 - \sigma_q)^2 + t_q^2)} = \\
&= \sum_{n=0}^{\infty} \frac{t_q(1 - 2\sigma_q)(4n + 1)}{((2n + \sigma_q)^2 + t_q^2)((2n + 1 - \sigma_q)^2 + t_q^2)} = \\
&= (1 - 2\sigma_q) \sum_{n=0}^{\infty} \frac{t_q(4n + 1)}{((2n + \sigma_q)^2 + t_q^2)((2n + 1 - \sigma_q)^2 + t_q^2)} = 0.
\end{aligned}$$

Sum

$$\sum_{n=0}^{\infty} \frac{t_q(4n+1)}{((2n+\sigma_q)^2+t_q^2)((2n+1-\sigma_q)^2+t_q^2)}$$

exists and is not equal to 0 when $t_q \neq 0$ so the equality (24) is performed exclusively at

$$\sigma_q = \frac{1}{2}.$$

So, assuming that an arbitrary nontrivial root q of zeta functions belongs to the union $\mathcal{P}_1 \cup \mathcal{P}_2$ we found that it belongs only to \mathcal{P}_2 , i.e. $\mathcal{P}_1 = \emptyset$.

And according to the fact that $\mathcal{P}_1 = \emptyset \Leftrightarrow \mathcal{P}_3 = \emptyset$ we have:

$$\mathcal{P}_3 = \mathcal{P}_1 = \emptyset, \quad \mathcal{P} = \mathcal{P}_2.$$

This proves the basic statement and the assumption which had been made by Bernhard Riemann about of the real parts of the nontrivial zeros of zeta function.

REFERENCES:

- [1] Abramowitz, M. and Stegun, I. A. (Eds.). "Polygamma Functions."§6.4 in Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, p. 260, 1972.
- [2] Allouche, J.-P. "Series and Infinite Products related to Binary Expansions of Integers."1992.
- [3] Davenport, H. Multiplicative Number Theory, 2nd ed. New York: Springer-Verlag, 1980.
- [4] Edwards, H. M. Riemann's Zeta Function. New York: Dover, 1974.
- [5] Karatsuba, A. A. and Voronin, S. M. The Riemann Zeta-Function. Hawthorn, NY: de Gruyter, 1992.
- [6] Keiper, J. B. "Power Series Expansions of Riemann's ξ Function."Math. Comput. 58, 765-773, 1992.
- [7] Knuth, D. E. The Art of Computer Programming, Vol. 1: Fundamental Algorithms, 3rd ed. Reading, MA: Addison-Wesley, 1997.
- [8] Lehmer, D. H. "The Sum of Like Powers of the Zeros of the Riemann Zeta Function."Math. Comput. 50, 265-273, 1988.
- [9] Smith, D. E. A Source Book in Mathematics. New York: Dover, 1994.
- [10] Titchmarsh, E. C. The Theory of the Riemann Zeta Function, 2nd ed. New York: Clarendon Press, 1987.
- [11] Valle-Poussin C.J. De la Recherches analytiques sur la theorie des wombres partie I,II,III, Am. Soc. Sci. Bruxelles, Ser. A. - 1896-1897, t. 20-21.
- [12] Voros, A. "Spectral Functions, Special Functions and the Selberg Zeta Function."Commun. Math. Phys. 110, 439-465, 1987.